Estimation of Soil Thermal Diffusivity and Heat Flux using Fractional Derivatives of Soil Temperature Variation

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Summary

• Soil temperature and heat flux at the ground surface are essential to evaluate the exchanges of energy and water vapor between land surfaces and the atmosphere. Appropriate methods are required to estimate the surface temperature and the heat flux based on underground temperature data easily available.

• We investigated the characteristics of the fractional derivatives of soil temperature variation. The semi-derivatives (half order derivatives) of homogeneous soils were almost linearly related to depth-directional gradients. The proportionality constant agreed well with the square root of the soil thermal diffusivity. Initial condition had limited effects only in a short "memory length". Therefore, time-dependent diffusivity in situ was evaluated if temperature gradients at the same depth were measured synchronously.

• Usually, thermal diffusivity in a drying surface layer increases with depth. Fractional derivatives of lower than half order would be appropriate to evaluate the thermal properties of such a layer.

• For the estimation of ground surface temperature and heat flux at the surface, numerical deconvolution of discrete soil temperature data using Green's function was confirmed to be efficient.

• Using the semi-derivatives, prediction of ground surface temperature was feasible when some surface temperature values up to now and reliable weather data after that were available.

Objectives

• To clarify the characteristics of the fractional derivatives of soil temperature variation for the estimation of soil thermal properties including depth-dependence.

• To solve the inverse problems of estimating the surface temperature and the heat flux based on soil temperature data in the ground.

Fractional Derivatives

• Spatial gradients of temperature $T(z, t)$ subjected to the diffusion equation with a constant diffusivity κ and an initially uniform condition are:

$$-\sqrt{\kappa}\,\frac{\partial T(z,t)}{\partial z} = \frac{1}{\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{t-u}}\,\frac{\partial T(z,u)}{\partial u}\,du \qquad (1)$$

The right-hand side of the equation is the special form of general fractional derivatives of order γ:

$$\frac{d^\gamma}{dt^\gamma} f(t) = \frac{1}{\Gamma(1-\gamma)}\int_0^t \frac{f'(u)}{(t-u)^\gamma}\,du \qquad (2)$$

Eq.1 means the depth-directional gradient is in proportion with the semi-derivatives (half order derivatives) of the temporal variation.
• The backward difference given below is a useful approximation for a discrete time series with constant interval τ.

$$\frac{d^\gamma}{dt^\gamma} f(t) \approx \tau^{-\gamma}\sum_{j=0} \frac{\Gamma(-\gamma+j)}{\Gamma(-\gamma)\Gamma(j+1)} f(t-j\tau)$$
$$\approx \tau^{-\gamma}\left(f(t) - \gamma f(t-\tau) - \frac{\gamma(1-\gamma)}{2!} f(t-2\tau) - \cdots \right) \qquad (3)$$

Application to Harmonic Waves

• A stationary harmonic fluctuation whose period was 24h was considered. The thermal conductivity λ and the volumetric heat capacity C were assumed to have depth-dependence.

$$T(z_0,t) = T_0\cos(\omega t + \phi)$$
$$\lambda = \lambda_0(1 + \alpha(z - z_0))$$
$$C = C_0(1 + \beta(z - z_0))$$

$$T_0 = 1$$
$$\lambda_0 = 1500\,Jm^{-1}hr^{-1}K^{-1}$$
$$C_0 = 1.30\times10^6\,Jm^{-3}K^{-1}$$

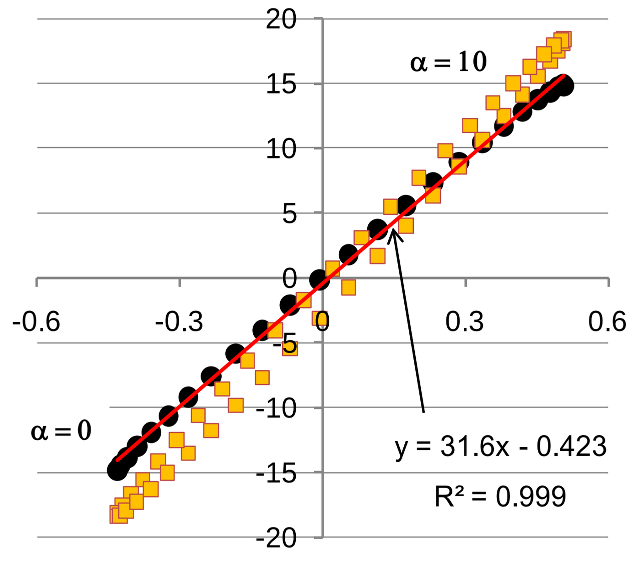


Fig. 1 Temperature Gradients at $z=z_0$ (Km^{-1}, vertical) ~ Sub-derivatives ($Kh^{-1/2}$, horizontal). α=β=0 and 10.

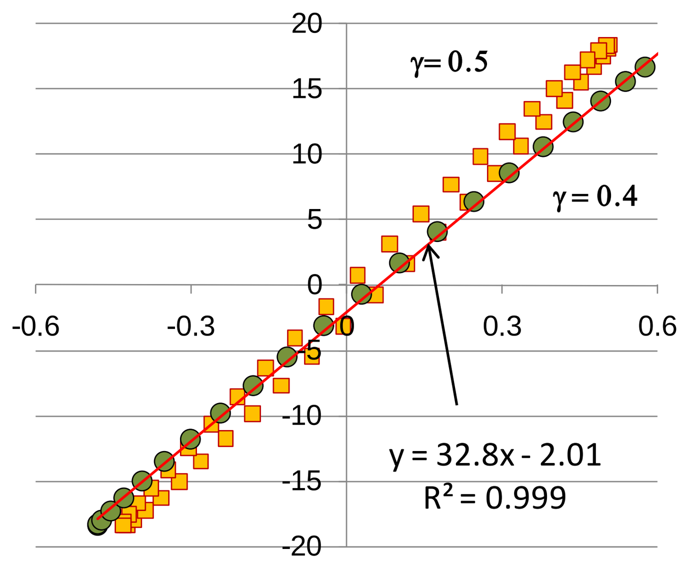


Fig. 2 Temperature Gradients at $z=z_0$ (Km^{-1}: vertical) ~ Fractional derivatives ($Kh^{-\gamma}$, horizontal). α=β=10.

Temperature and Heat Flux at the Ground Surface

• According to Duhamel's theorem, soil temperature $T(z, t)$ is associated with the ground surface temperature $T_0(t)$ or the surface heat flux $q_0(t)$. If $g(t)$ represents these functions,

$$T(z,t) = \int_0^t \psi(z,t-u)\cdot g'(u)\,du \approx \sum_{j=0}^{n-1}\psi\left(z,(n-j-\tfrac{1}{2})\tau\right)\cdot \Delta g\left((j+\tfrac{1}{2})\tau\right) \qquad (4)$$

$$\text{where, } \Delta g\left((j+\tfrac{1}{2})\tau\right) = g((j+1)\tau) - g(j\tau), \qquad g(0) = 0.$$

$$\text{for } g(t) = T_0(t): \qquad \psi(z,t) = erfc\left(z/2\sqrt{\kappa t}\right),$$

$$\text{for } g(t) = q_0(t): \qquad \psi(z,t) = \frac{2\sqrt{t}}{C\sqrt{\kappa}}\,ierfc\left(z/2\sqrt{\kappa t}\right).$$

• Eq.4 indicates the deconvolution of a series $T(z, j\tau)$ with $\Psi(z, (j+1/2)\tau)$ gives a series of $\Delta g((j+1/2)\tau)$. The series of $g(j\tau)$ is obtained by adding the Δg successively.

Application to Soil Temperature

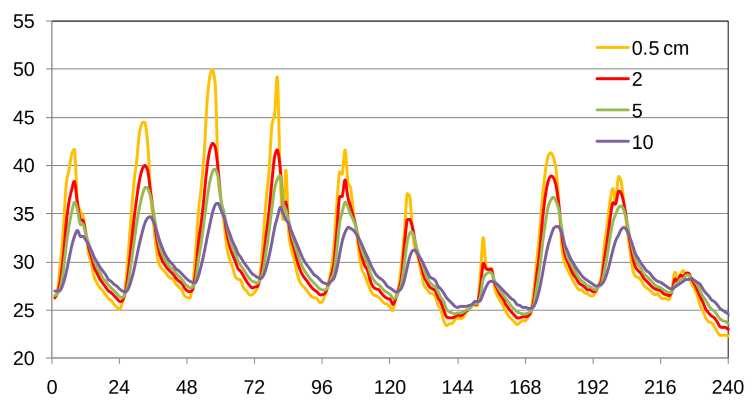


Fig.3 Soil temperature data analyzed (Mie University Farm). The horizontal axis represents elapsed hours from 6:00 am Aug.3, 2003.

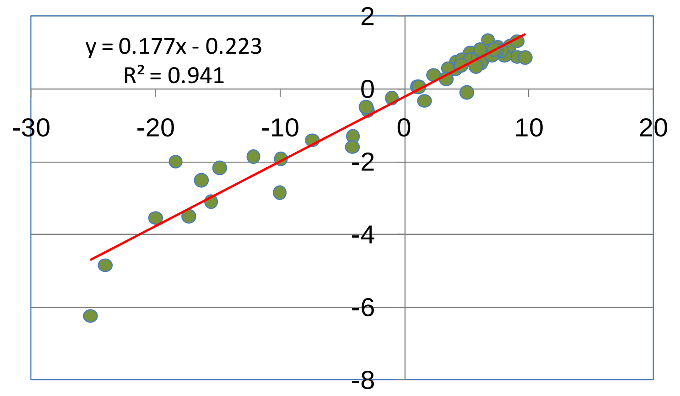


Fig.4 Temperature gradients at z=0.5cm (Kcm^{-1},vertical) ~ Sub-derivatives ($Kh^{-1/2}$, horizontal). Aug. 3 ~ 5. The thermal diffusivity was obtained as $k=8.87\times10^{-3}$ cm^2/s.

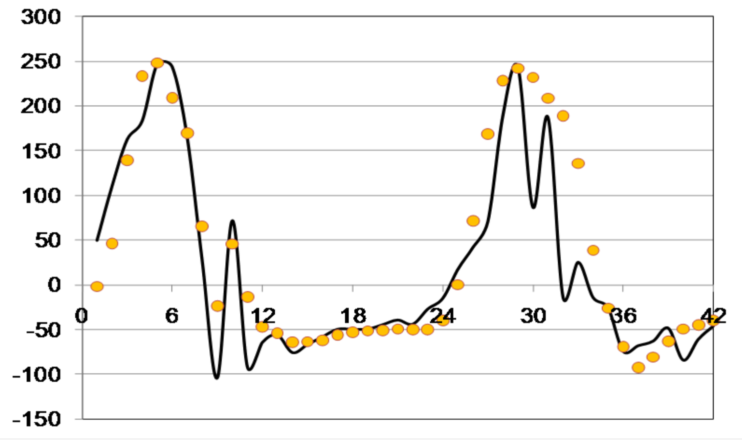


Fig.5 Surface heat flux estimated by deconvolution of the soil temperature of the depth z=2cm (dots). The curve is a heat flux intensity (W/m^2) measured at the same depth.

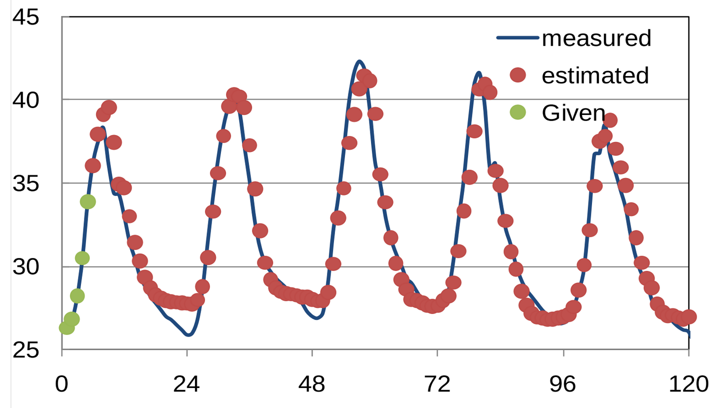


Fig.6 Prediction of soil temperature variation at the depth z=2cm using 5 initial values and heat flux data measured at the same depth by two heat flux meters. Aug. 3~7.

Prediction of Surface Temperature

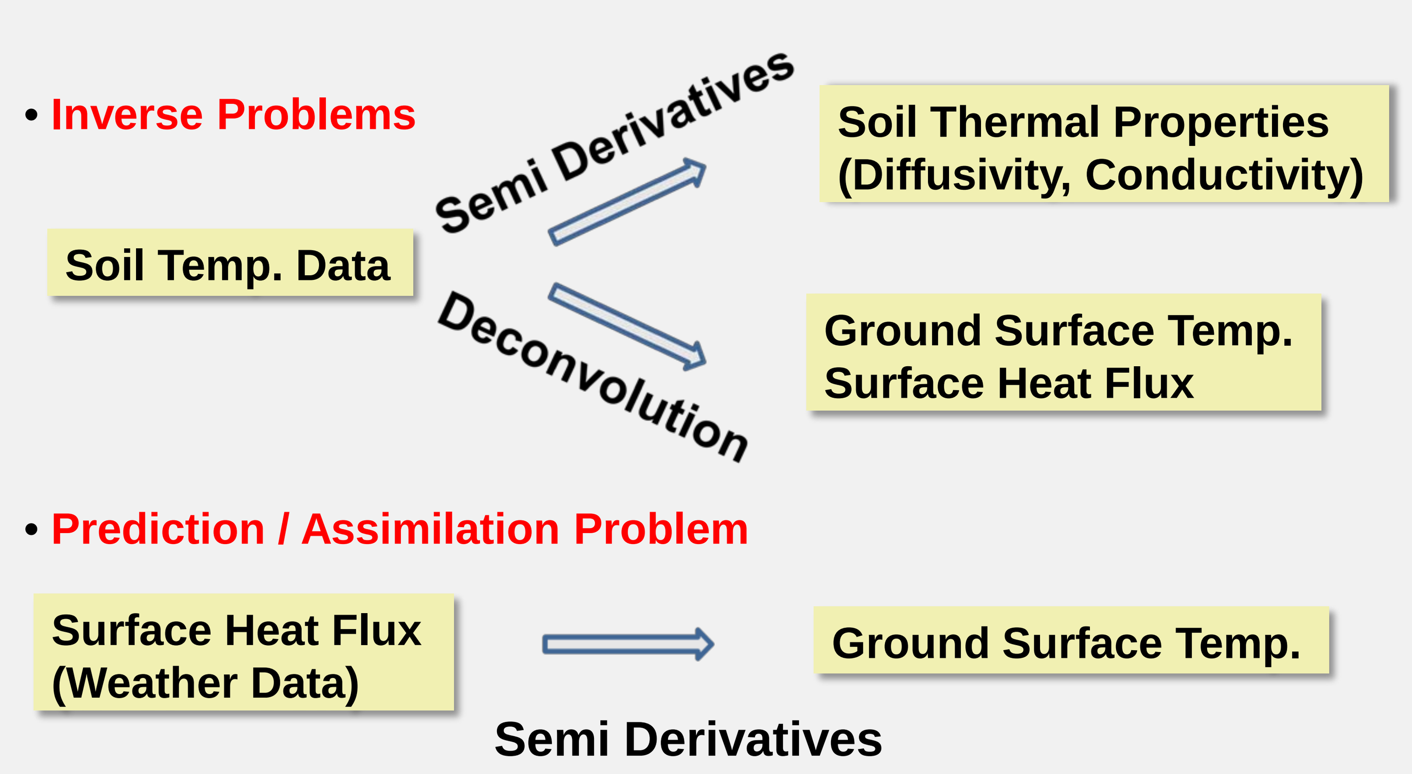
• Using eq.3, the ground surface temperature T_0(t) is inferred based on a series of the surface temperature before: T_0(t-τ), T_0(t-2τ), T_0(t-3τ), ··· , and the heat flux into the soil q_0(t).

$$T_0(t) = \frac{\tau^{1/2}}{C\sqrt{\kappa}}q_0(t) + \frac{1}{2}T_0(t-\tau) + \frac{1}{2!2^2}T_0(t-2\tau) + \frac{1\cdot3}{2!2^3}T_0(t-3\tau) + \cdots \qquad (5)$$

• An example of the prediction of soil temperature variation is given in Fig. 6. The method should be superior to the classical "force-restore method" frequently used in GCM.

Conclusions

• The fractional-order derivatives and deconvolution technique make it possible to find new, effective non-conventional solutions to important technological problems.



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